Advanced Digital Signal Processing
Sidelobe Canceller
(Beam Former)

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Sidelobe canceller (Beam Former):

A complete analysis/discussion of results is included.
Assumptions:

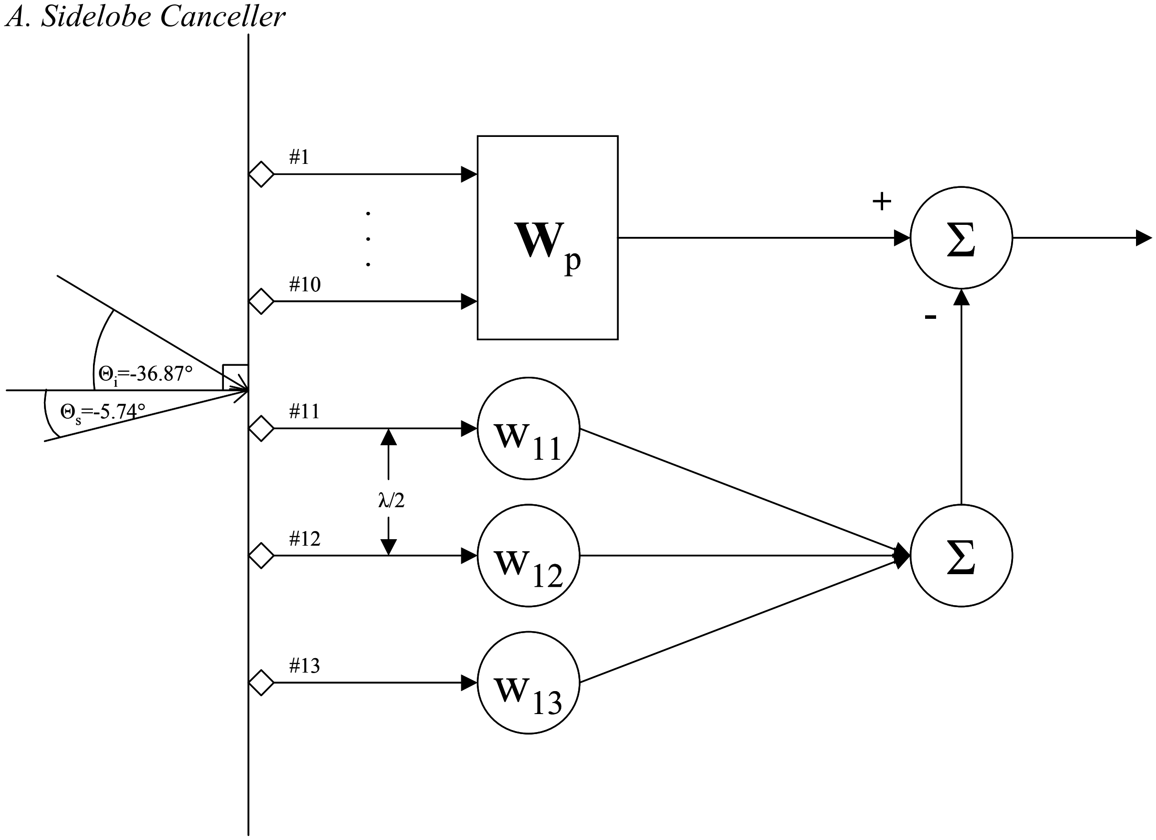
1. A linear equally spaced array of thirteen sensors spaced by one half wavelength.
2. Plane wave propagation.
3. All signals/interference are narrowband. (Assuming for convenience a normalized frequency of π.)
4. All arrival angles are defined with respect to the line perpendicular to the axis of the sensor array.

The beampattern is defined as $10\log_{10}(|r(\theta,\pi)|^2)$, where $r(\theta,\pi) = \mathbf{w}^H\mathbf{d}(\theta,\pi)$. Here **w** is the vector of beamformer weights and $\mathbf{d}(\theta,\pi)$ is the steering vector or array response vector for direction θ and frequency π .Assume the data consists of a signal arriving from $\theta_s = -5.74$ degrees with power σ_s^2, an interferer arriving from $\theta_I = 36.87$ degrees with power σ_I^2,and uncorrelated (white) noise of power σ_n^2. The signal, interferer, and noise are all statistically independent.

The project is organized according to problem number, e.g., 1, 2a, 2b, etc. For each, a summary that highlights the results obtained and contains discussion comments is included.

Problem Statements:

A. Sidelobe Canceller



1. Assume the first 10 sensors are used to form the primary antenna output. Find the weight vector (**w**p) of minimum norm ($\mathbf{w}_p^H \mathbf{w}_p$) that has unit response to signals arriving from θ_s degrees ($r(\theta_s,\pi) = 1$). Plot the beampattern for **w**p.

2. Assume $\sigma_I^2 \approx 0$ and that an adaptive weight is placed in sensor channel 11 (12-13 are set to zero). Find an expression for the gain to the signal $10\log_{10}(|r(\theta_s,\pi)|^2)$, as a function of σ_s^2/σ_n^2. Plot this expression over a range of of σ_s^2/σ_n^2 from 10^{-3} to 10^3. Plot beampatterns for $\sigma_s^2/\sigma_n^2 = 10^{-2}, 1, 10^2$.

3. Assume $\sigma_s^2 \approx 0$. Find an expression for the gain to the interferer, $10\log_{10}(|r(\theta_I,\pi)|^2)$ as a function of σ_I^2/σ_n^2 and the number of adaptive weights (either 1, 2, or 3 weights with unused weights set equal to zero). Plot this expression over a range of σ_I^2/σ_n^2 from 10^{-3} to 10^5 for 1, 2, and 3 adaptive weights. Also plot the interference output power, $10\log_{10}(\sigma_I^2|r(\theta_I,\pi)|^2)$, as a function of σ_I^2 (over a range 10^{-3} to 10^5, let $\sigma_n^2 = 1$). Plot beampatterns for $\sigma_I^2/\sigma_n^2 = 10^{-2}, 1, 10^2, 10^4$ for a single adaptive weight. Do multiple weights provide significant increases in interference cancellation relative to a single weight?

4. Assume $\sigma_I^2/\sigma_n^2 = 10^4$. Plot the gain to the signal, $10\log_{10}(|r(\theta_s,\pi)|^2)$, as a function of σ_s^2/σ_n^2 (over a range 10^{-3} to 10^3) for one (channel 11), two (channels 11,12), and three (channels 11-13) adaptive weights. Comment.

B. Generalized Sidelobe Canceller

We desire a beamformer that minimizes the output power subject to the signal response constraint $r(\theta_s,\pi) = 1$. Utilize all thirteen sensors.

1. Determine $\mathbf{w}_q$ and $\mathbf{C}_n$ in the GSC representation $\mathbf{w} = \mathbf{w}_q - \mathbf{C}_n\mathbf{w}_n$. Plot the beampattern of $\mathbf{w}_q$. Also plot the beampatterns of each column of $\mathbf{C}_n$ (all on the same graph).

2. Consider replacing $\mathbf{C}_n$ by $\mathbf{C}_{nT} = \mathbf{C}_n\mathbf{T}$ where $\mathbf{T}$ is a nonsingular matrix. Show that **w** is independent of **T** for any data covariance matrix **R**. Comment.

3. Assume $\sigma_I^2/\sigma_n^2 = 10^4$. Find an expression for the gain to the signal, $10\log_{10}(|r(\theta_s,\pi)|^2)$, as a function of σ_s^2/σ_n^2. Plot this function.

4. Find an expression for the gain to the interferer, $10\log_{10}(|r(\theta_I,\pi)|^2)$,as a function of σ_I^2/σ_n^2 . Plot this expression over a range of σ_I^2/σ_n^2 from 10^{-3} to 10^5. Note that:

$$(A + uv^H)^{-1} = A^{-1} - \frac{(A^{-1}u)(v^H A^{-1})}{1 + v^H A^{-1}u}$$

where all matrices are assumed nonsingular, **u** and **v** are column vectors.

5. Assume $\sigma_I^2/\sigma_n^2 = 10^4$ and that the signal is arriving from 4 degrees instead of θ_s (not changing the constraints to this direction). Evaluate and plot the gain to the signal, $10\log_{10}(|r(4,\pi)|^2)$, as a function of σ_s^2/σ_n^2 (over a range 10^{-3} to 10^3). Plot the beampattern for $\sigma_s^2/\sigma_n^2 = 10^2$. Comment.

6. Assume the data is due to a signal at θ_s degrees with power 10, uncorrelated noise with

power 1, and five interferers with the following directions and powers:
1) –40 degrees at 10^4
2) –30 degrees at 10^5
3) 20 degrees at 10^4
4) 45 degrees at 10^3
5) 70 degrees at 10^4
Plot the beampattern. Compute the array gain, defined as the ratio of the SNR at the beamformer output to the SNR at a single sensor (in this definition noise refers to both interference and uncorrelated noise).

PART A.1

Taps/(receivers) 1-10 are multiplied by a weight vector $\mathbf{w_p}$ to satisfy the constraint of passing desired signal $s(\Theta_s,\pi)$ (The signal from direction Θ_s at frequency π.) with unit gain.

The constraint is :

$$\sum_{k=0}^{M-1} w_k e^{-jk\phi_0} = g \quad \text{i.e.} \quad r(\Theta_s,\pi) = \sum_{k=0}^{9} w_k e^{-jk\phi_s} = 1$$

The Steering vector $d(\Theta_s,\pi)$ =

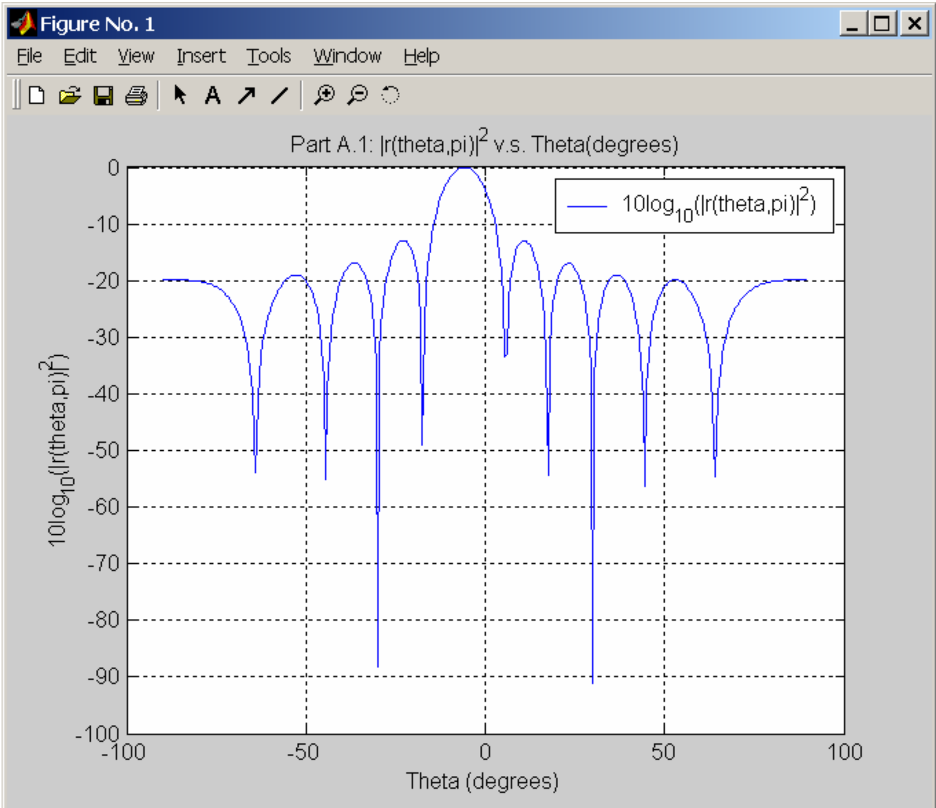
$$d(\Theta_s,\pi) = [1 \quad e^{-j\phi_s} \quad e^{-j2\phi_s} \quad e^{-j3\phi_s} \quad e^{-j4\phi_s} \quad e^{-j5\phi_s} \quad e^{-j6\phi_s} \quad e^{-j7\phi_s} \quad e^{-j8\phi_s} \quad e^{-j9\phi_s}]^T$$

where $\phi_s = \sin(\Theta_s)$ - the time delay the signal takes to get to an adjacent antenna after hitting one.

The tap weight vector wp was calculated using

$$w_p = \frac{g * R^{-1}\underline{d}(\Theta_s,\pi)}{\underline{d}^H(\Theta_s,\pi)R^{-1}\underline{d}(\Theta_s,\pi)}$$ With R = 1 since due to white noise only on the inputs without the signal of interest. This yields $w_p = \underline{d}(\Theta_s,\pi)(\underline{d}^H(\Theta_s,\pi)\underline{d}(\Theta_s,\pi))^{-1} = \frac{\underline{d}(\Theta_s,\pi)}{10}$

Plotting $10\log_{10}(|r(\Theta_s,\pi)|^2)$ where $r(\Theta_s,\pi) = \underline{\mathbf{w}}_p^H \underline{\mathbf{d}}(\Theta_s,\pi)$



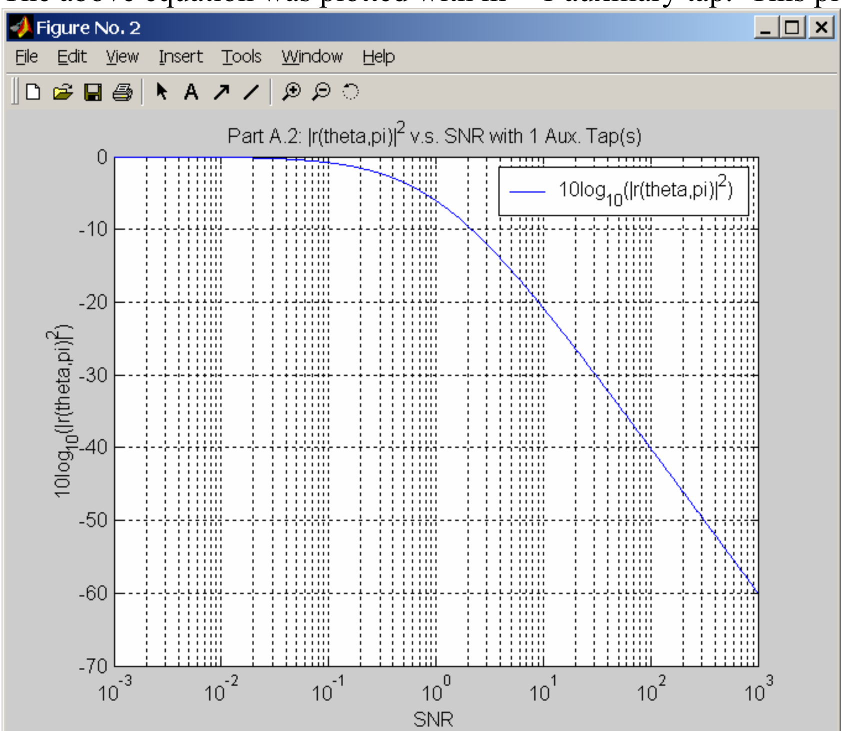
Notice that the signal gain at $\Theta_{s=-5.75}$ is 1. I.e. it passes the desired signal without distortion. This plot was done by a_1.m

PART A.2

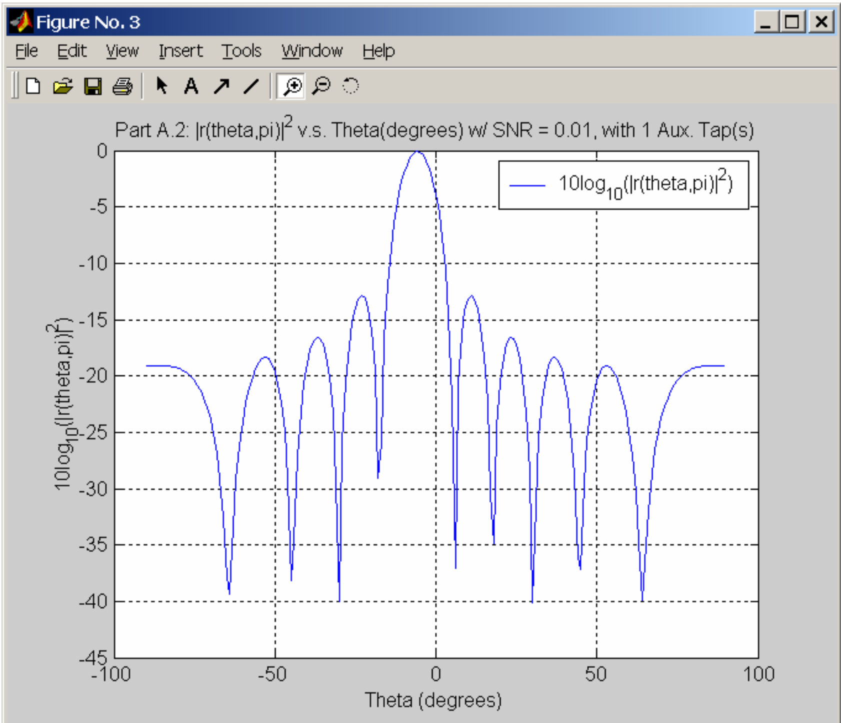
The signal gain as a function of signal to noise power was found.

$$r(\Theta_s, \pi) = 1 - \frac{\sigma_s^2}{\sigma_n^2} m + \frac{\left(\frac{\sigma_s^2}{\sigma_n^2}\right)^2}{1 - \frac{\sigma_s^2}{\sigma_n^2} m}$$ Where m is the number of auxiliary taps.

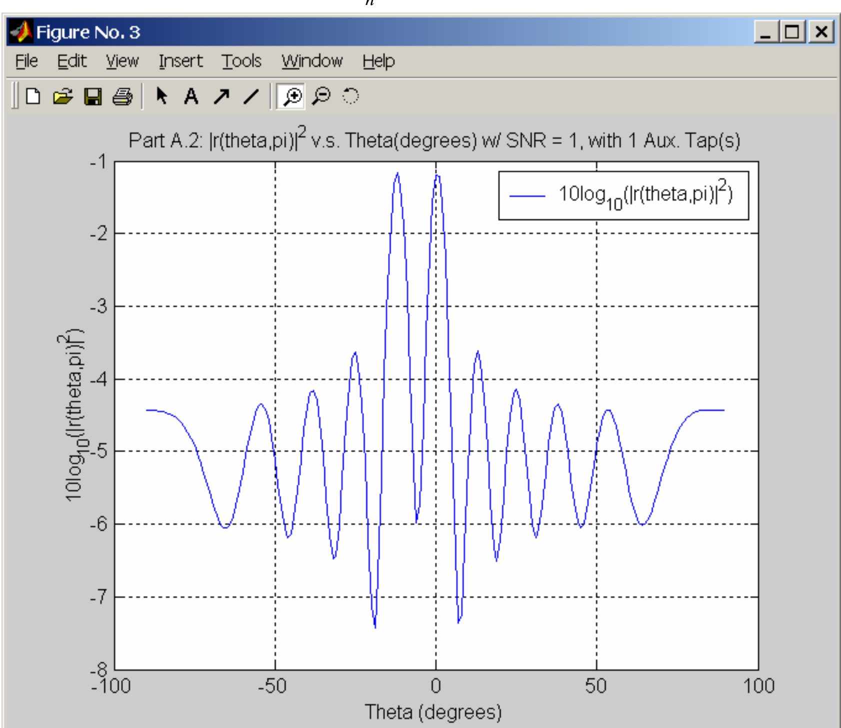
The above equation was plotted with m = 1 auxiliary tap. This plot is shown below:



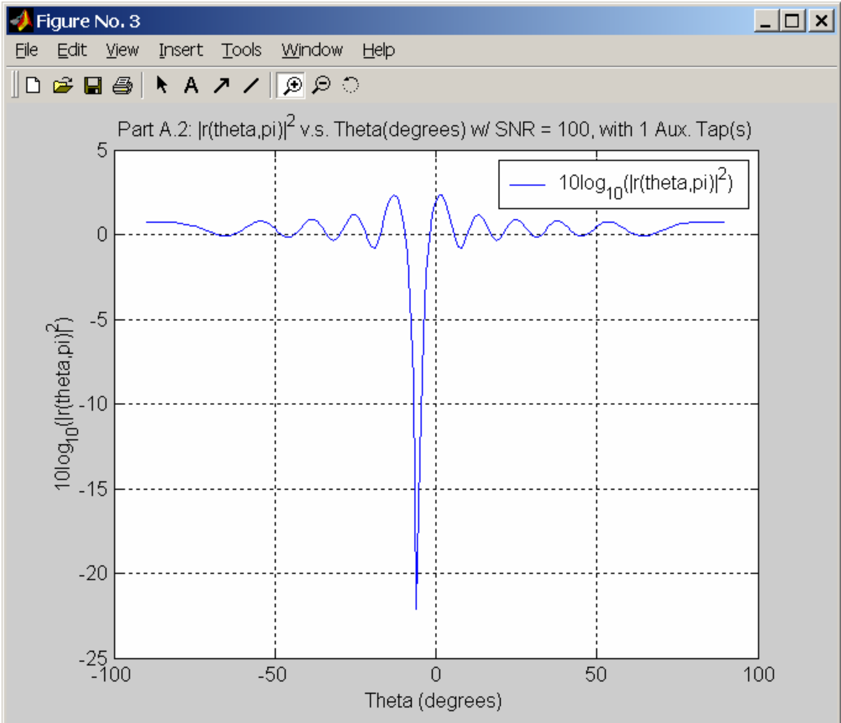
The beam pattern plot for $\frac{\sigma_s^2}{\sigma_n^2} = 0.01$ is shown below:



The beam pattern plot for $\frac{\sigma_s^{\ 2}}{\sigma_n^{\ 2}} = 1.0$ is shown below:



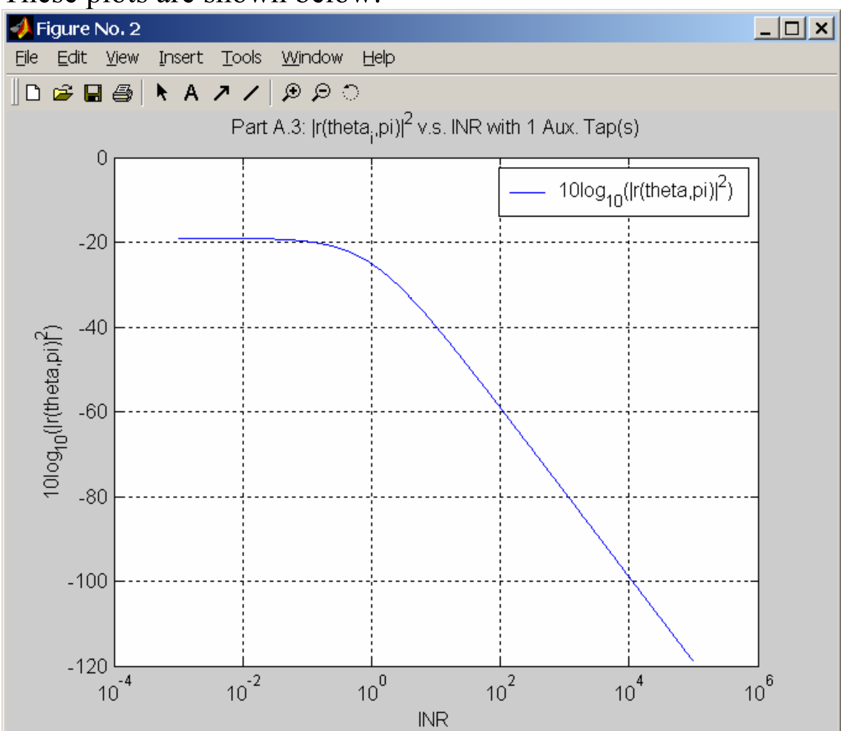
The beam pattern plot for $\frac{\sigma_s^{\ 2}}{\sigma_n^{\ 2}} = 100.0$ is shown below:

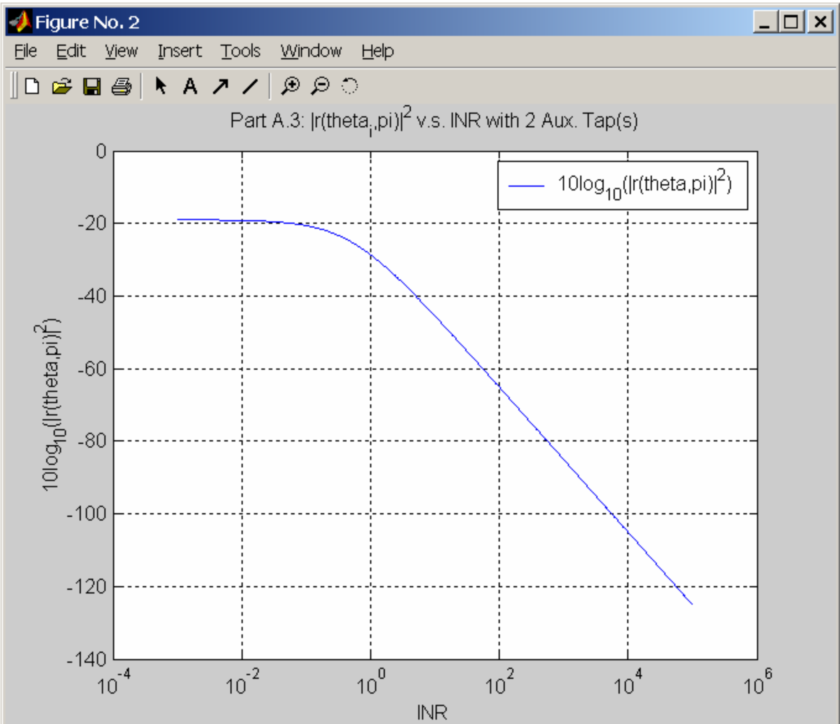


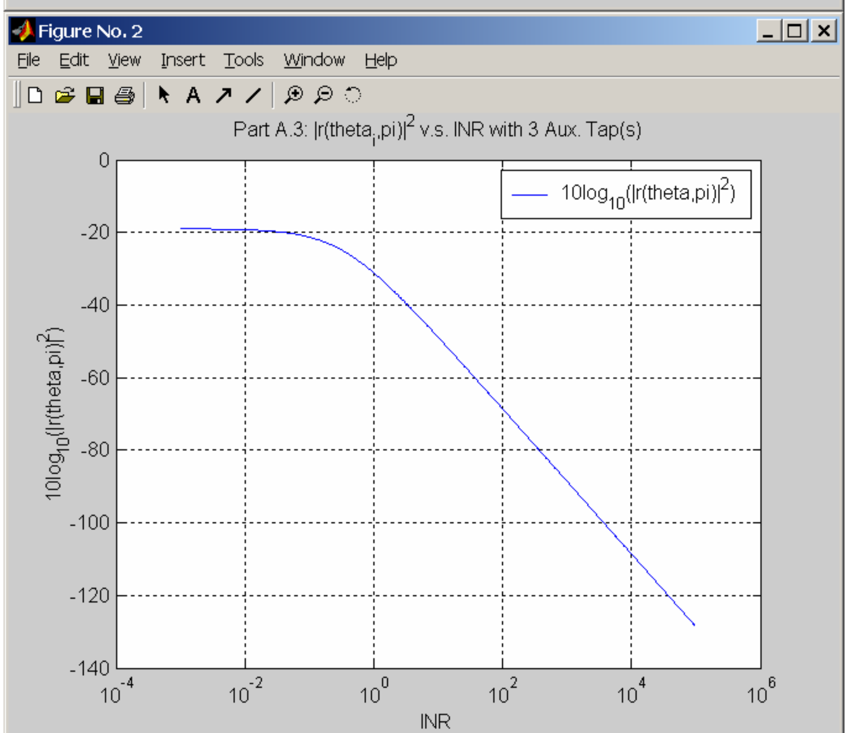
It becomes apparent as the signal to noise ratio gets better more signal of interest leaks through the auxiliary taps causing the Weiner filter to kill the signal itself in order to minimize the mean squared error. These plots were done by a_2.m

PART A.3

An expression was derived for the gain of the interferer in terms of the interference power to noise power ratio $\frac{\sigma_i^2}{\sigma_n^2}$: $r(\Theta_i, \pi) = 1 - \frac{\sigma_i^2}{\sigma_n^2} m + \frac{\left(\frac{\sigma_i^2}{\sigma_n^2}\right)^2}{1 - \frac{\sigma_i^2}{\sigma_n^2} m}$ where m is the number of auxiliary taps. This gain was plotted over the range of $10^{-3} < \frac{\sigma_i^2}{\sigma_n^2} < 10^5$ for 1, 2, and 3 auxiliary taps. These plots are shown below:







It is worthwhile to note that the interference gain starts dropping at a lower INR as the number of auxiliary taps increases.

The interference power, $10\log_{10}(\sigma_i^2 \, |r(\Theta_i,\pi)|^2)$ was plotted as a function of the interference to noise ration $\frac{\sigma_i^2}{\sigma_n^2}$ over the range of $10^{-3} < \frac{\sigma_i^2}{\sigma_n^2} < 10^5$ for 1, 2, and 3 auxiliary taps. These plots follow below:

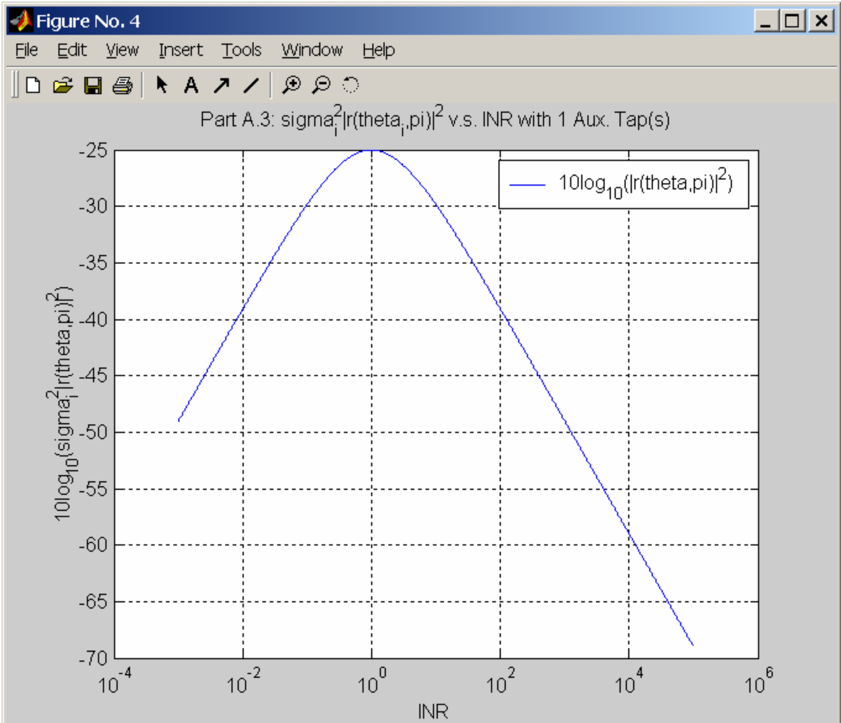


Figure No. 4
File Edit View Insert Tools Window Help
Part A.3: sigma_i^2|r(theta_i,pi)|^2 v.s. INR with 1 Aux. Tap(s)
10log10(|r(theta,pi)|^2)
10log10(sigma_i^2|r(theta,pi)|^2)
INR
-25
-30
-35
-40
-45
-50
-55
-60
-65
-70
10^-4
10^-2
10^0
10^2
10^4
10^6

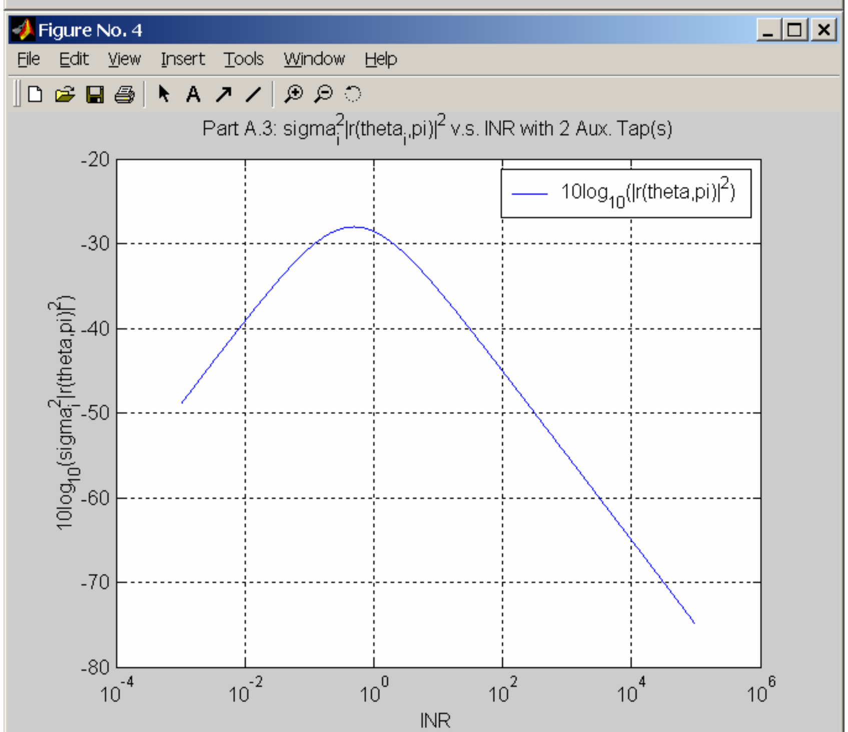


Figure No. 4
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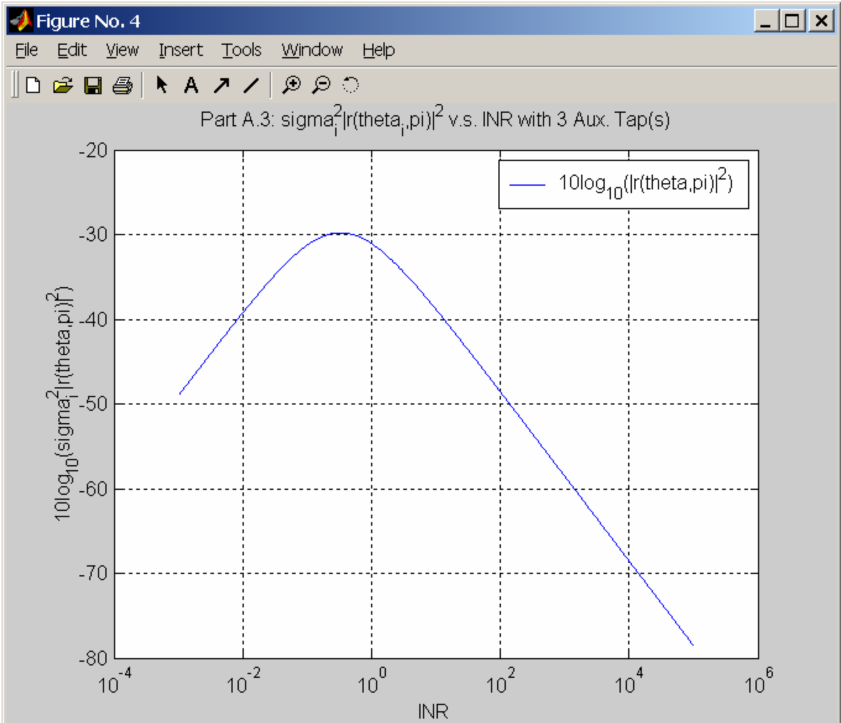
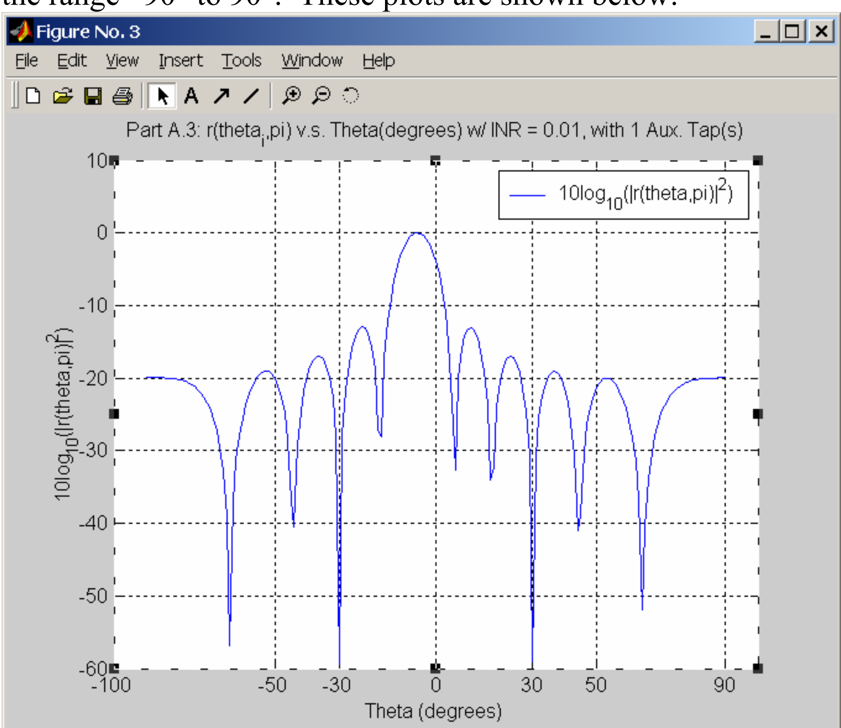
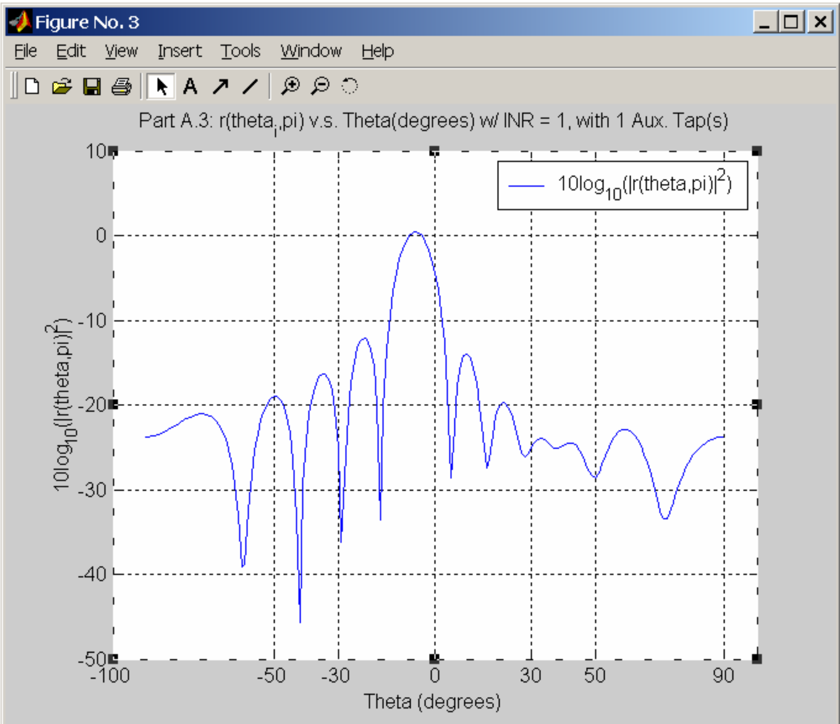
Part A.3: sigma_i^2|r(theta_i,pi)|^2 v.s. INR with 2 Aux. Tap(s)
10log10(|r(theta,pi)|^2)
10log10(sigma_i^2|r(theta,pi)|^2)
INR
-20
-30
-40
-50
-60
-70
-80
10^-4
10^-2
10^0
10^2
10^4
10^6



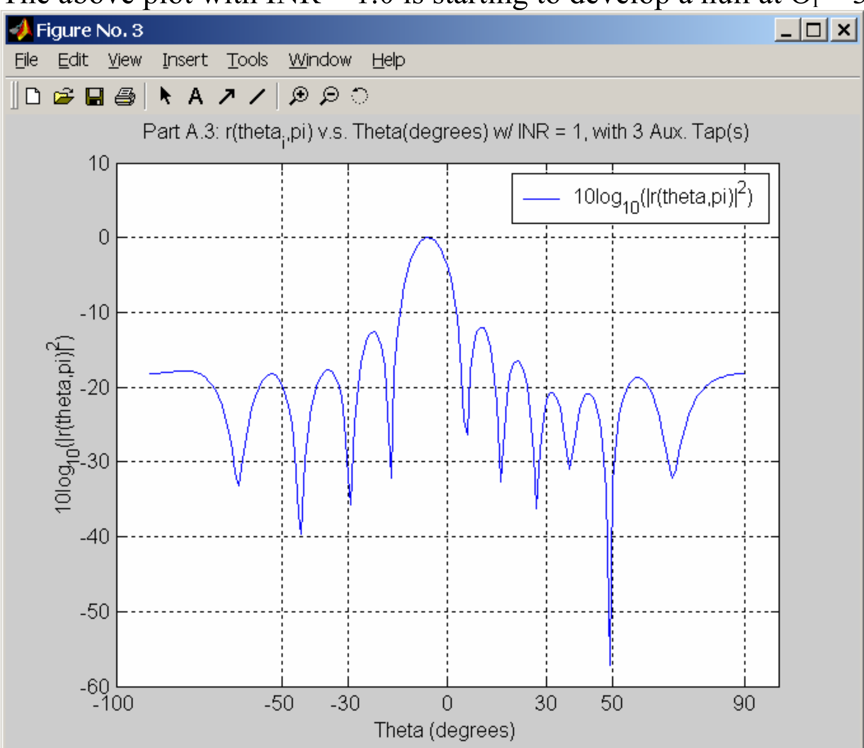
The beam patterns for $\frac{\sigma_i^{\ 2}}{\sigma_n^{\ 2}} = 10^{-2}, 1, 10^2, 10^4$ were plotted by sweeping the steering vector over the range –90° to 90°. These plots are shown below:



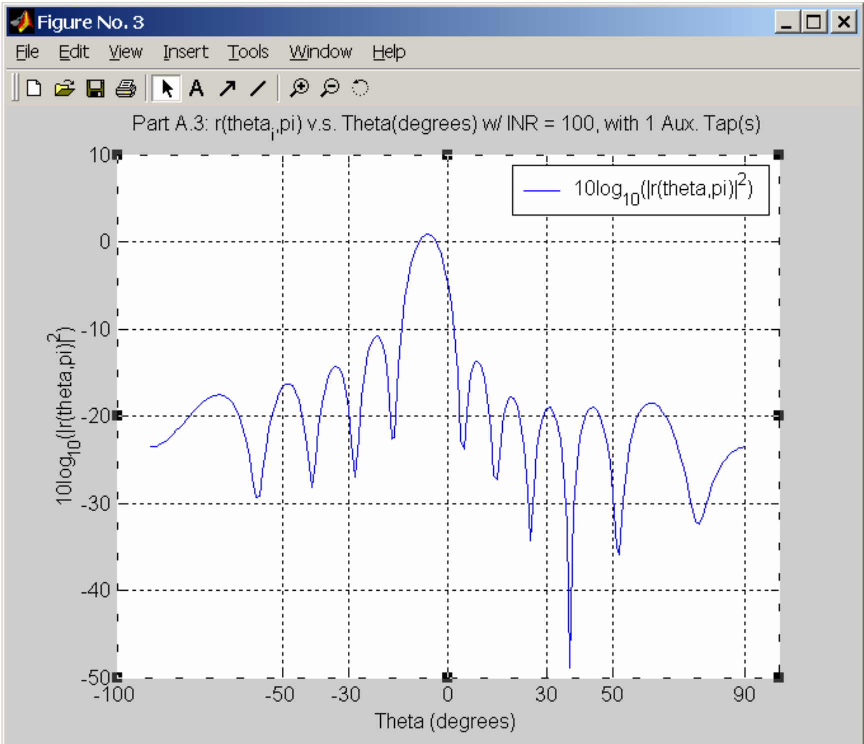
The above plot with INR = 0.01, does not have a null at $\Theta_i = 36.87°$.



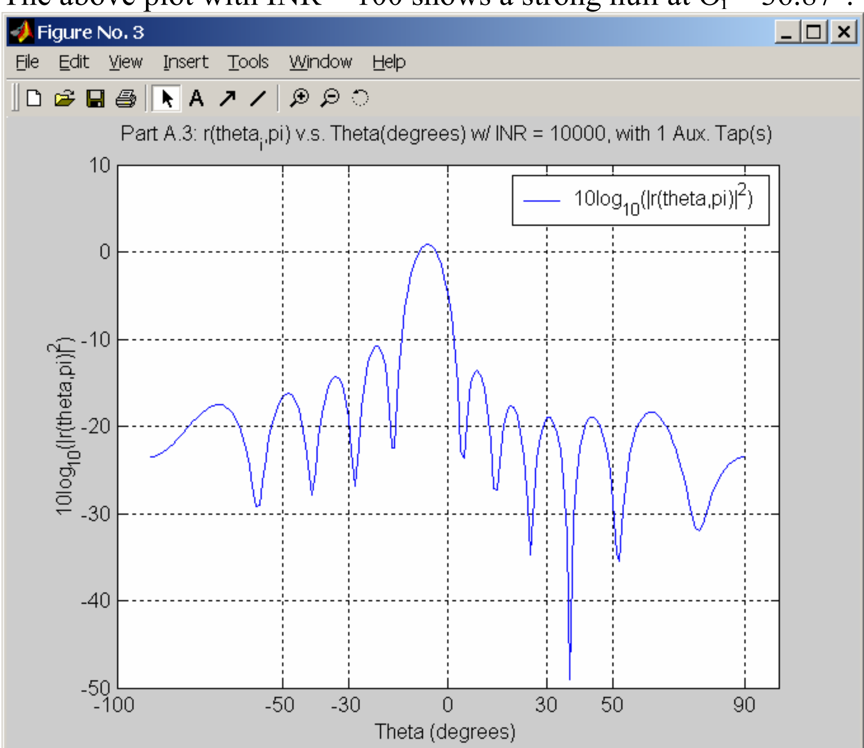
The above plot with INR = 1.0 is starting to develop a null at $\Theta_i = 36.87°$.



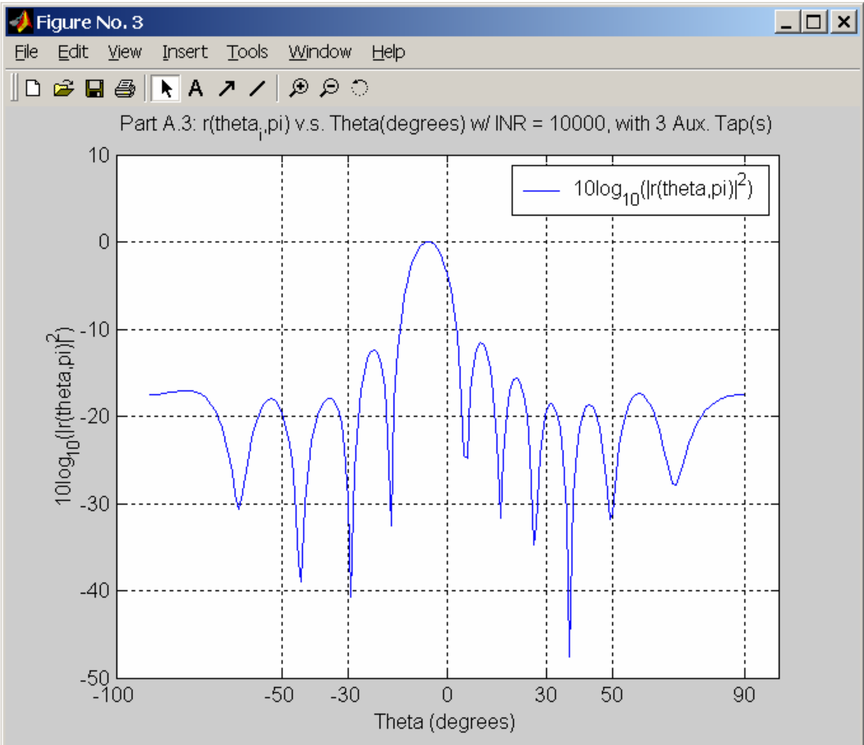
The above plot with INR = 1.0 shows a stronger null at $\Theta_i = 36.87°$ due to two additional auxiliary taps.



The above plot with INR = 100 shows a strong null at $\Theta_i = 36.87°$.



The above plot with INR = 10^4 shows a strong null at $\Theta_i = 36.87°$. However this null is not significantly stronger than the null generated when the INR is two orders of magnitude smaller.

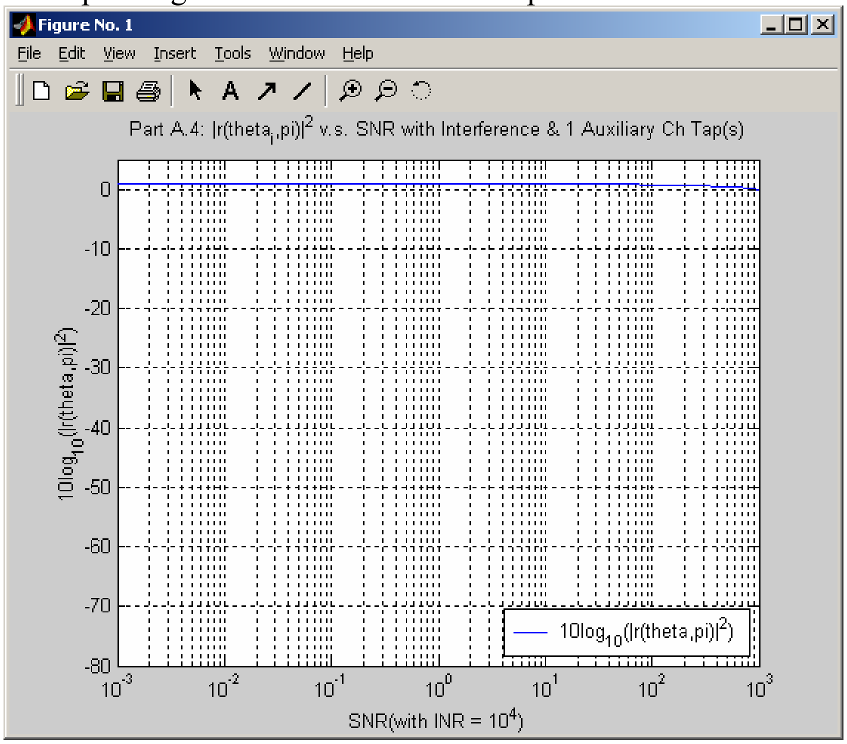


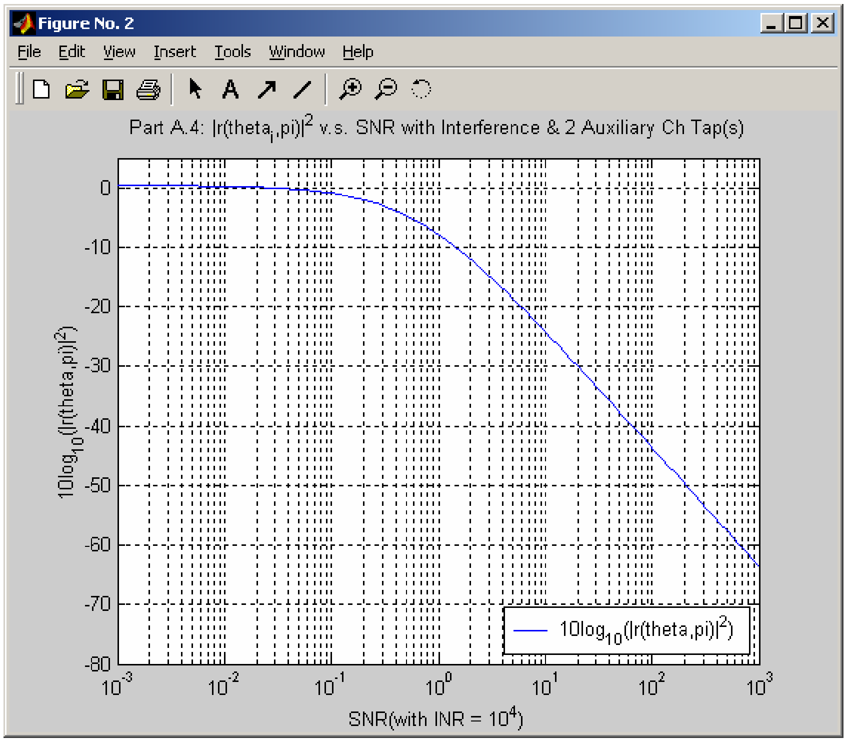
The above plot with INR = 10^4 shows a strong null at $\Theta_i = 36.87°$ with two additional taps. This null is not improved by adding the two additional taps.

In summary interference cancellation is related to both INR and the number of auxiliary taps. However it is most sensitive to the INR. The plots for part A.3 were generated by a_3.m.

PART A.4

With the interference to noise power ratio is fixed at 10^4 the signal gain $10\log_{10}(r(\Theta_s,\pi)|^2)$ was plotted while varying the signal to noise ratio over the range $10^{-3} < \frac{\sigma_s^2}{\sigma_n^2} < 10^3$ for the number of auxiliary taps varying from one to three taps. These plots follow below and are labeled corresponding to number of additional taps:





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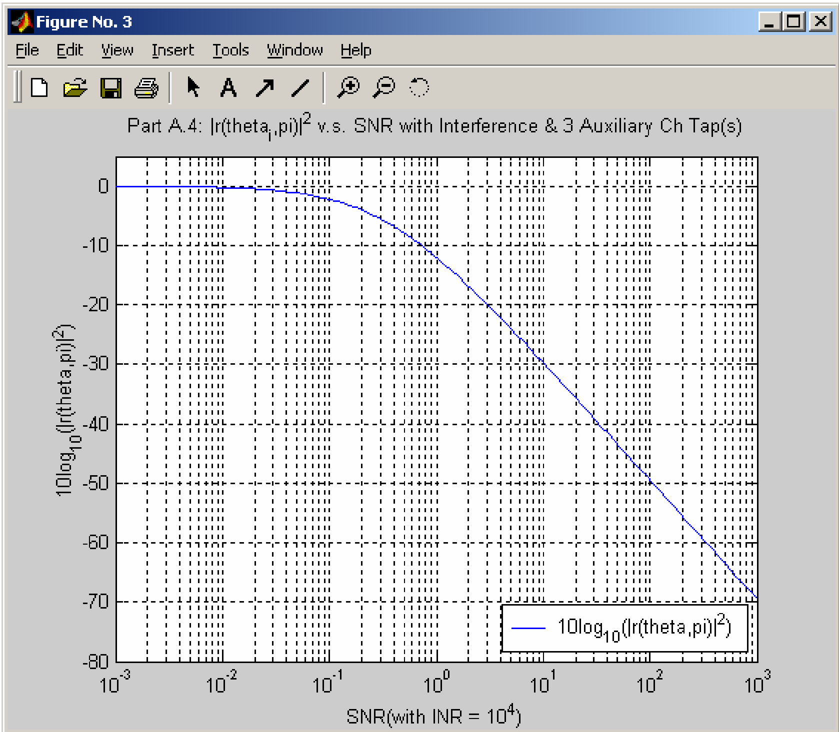
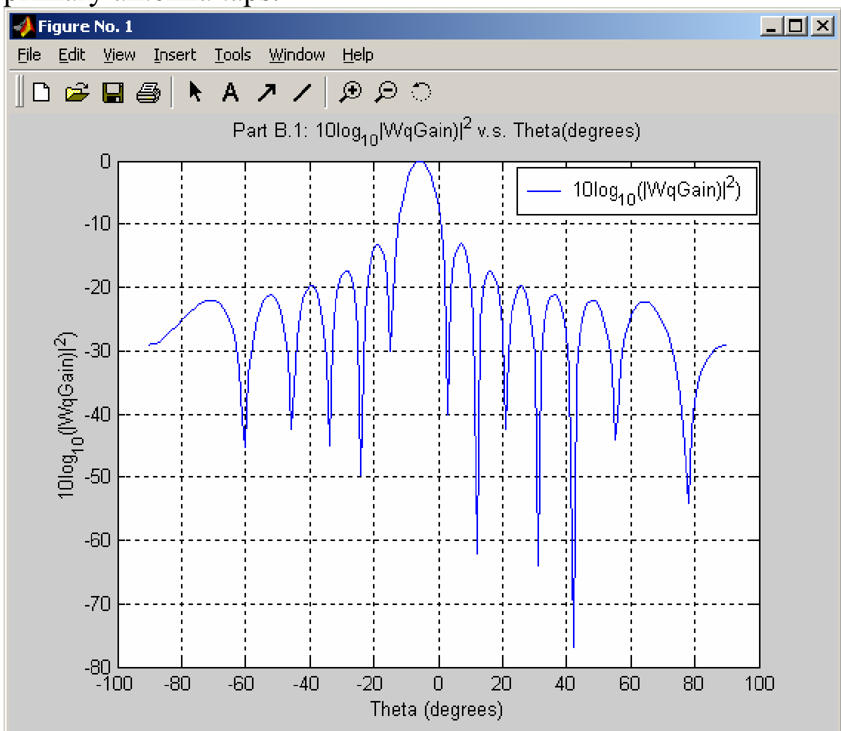


The same primary problem exists with the beamformer. As the number of auxiliary channels increases, the amount of desired signal leaking through the auxiliary channels increases. When this happens the filter does what it is supposed to: minimize mean squared error, which manifests by reducing the filter output by the auxiliary tap gains times the signal of interest. The plots for part A.4 were generated by a_4.m.

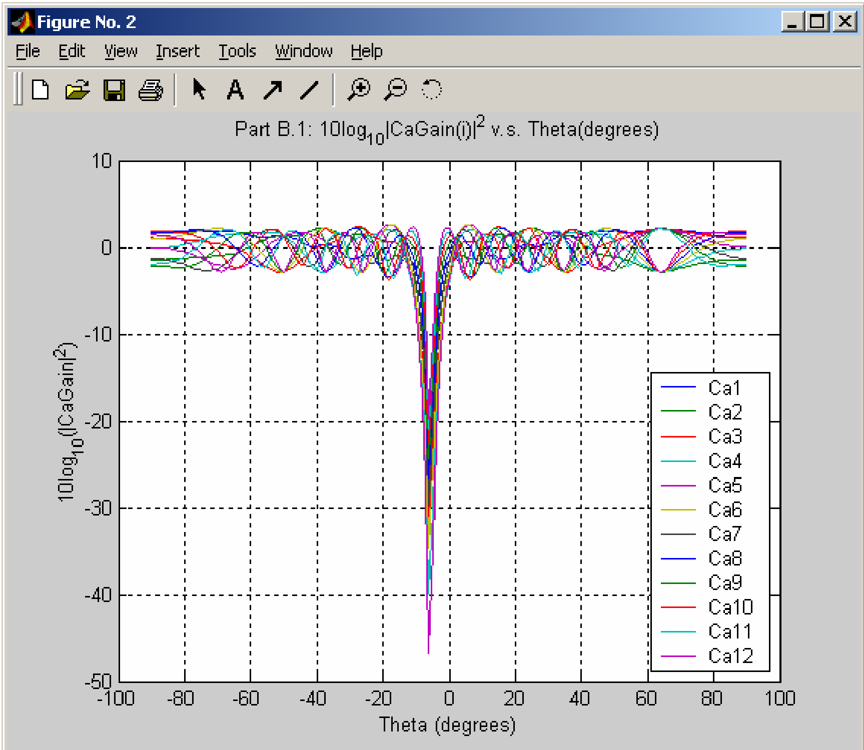
PART B.1

The constraint matrix C was determined. Using Matlab, I determined the orthogonal basis to C which is C_a as well as the auxiliary channel weights w_a. C_a can be interpreted as a Blocking FIR Filter Bank centered at $\Theta_{s = -5.75}$.

Shown below is the gain plot v.s. signal angle for w_q which is the bank of coefficients for the primary antenna taps.



The Ca Matrix gain was plotted for each column of Ca. This plot is shown below:



The previous two graphs were generated by the M-File: b_1.m

PART B.2

PART B.3

PART B.4

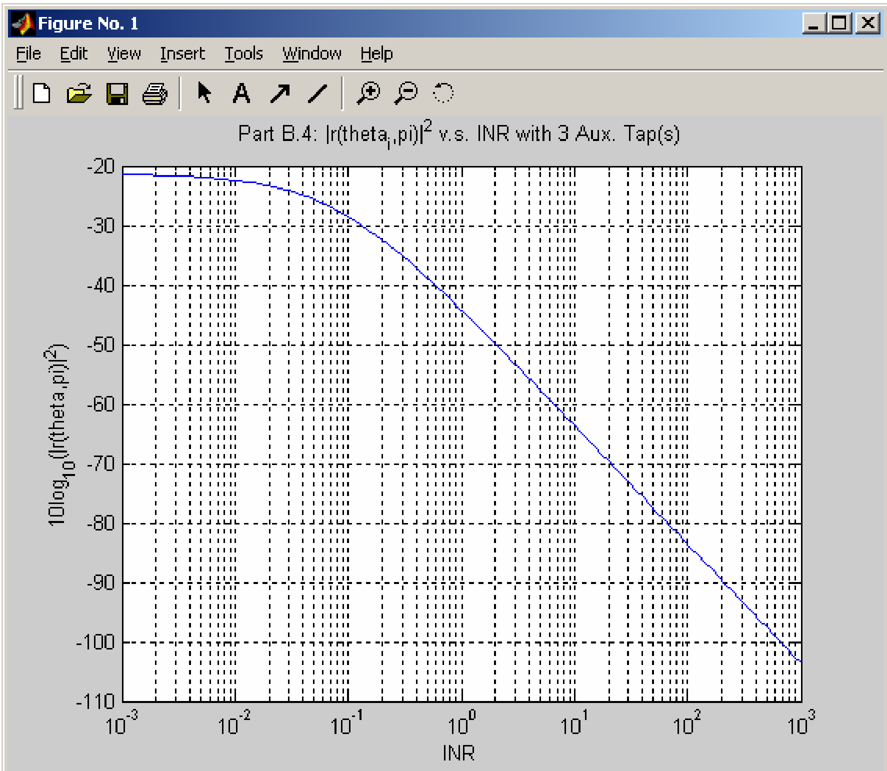
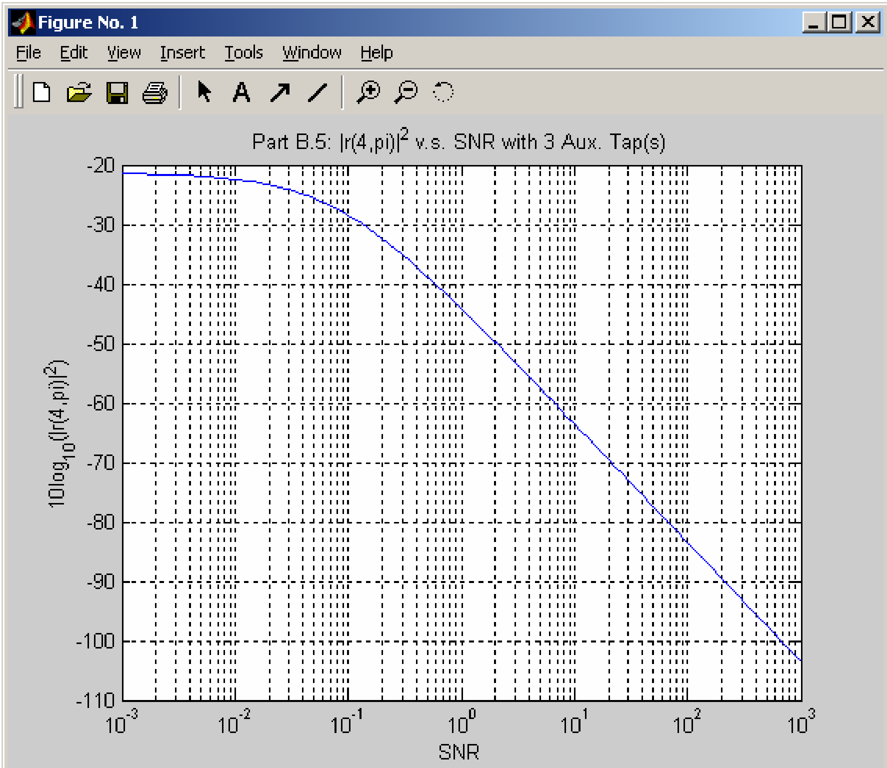
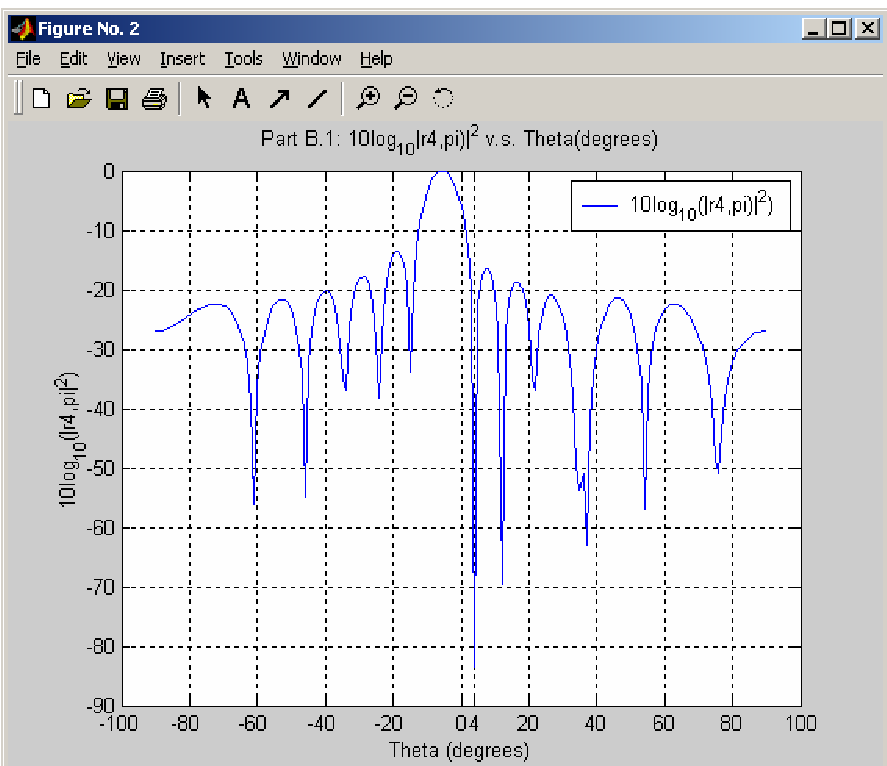


Figure No. 1
File Edit View Insert Tools Window Help
Part B.4: |r(theta1,pi)|2 v.s. INR with 3 Aux. Tap(s)
10log10(|r(theta,pi)|2)
-20
-30
-40
-50
-60
-70
-80
-90
-100
-110
10-3
10-2
10-1
100
101
102
103
INR

PART B.5





The signal appears at a different direction from where it is expected. The Filter treats the signal at four degrees as interference and generates a null in that direction.

PARTB.6

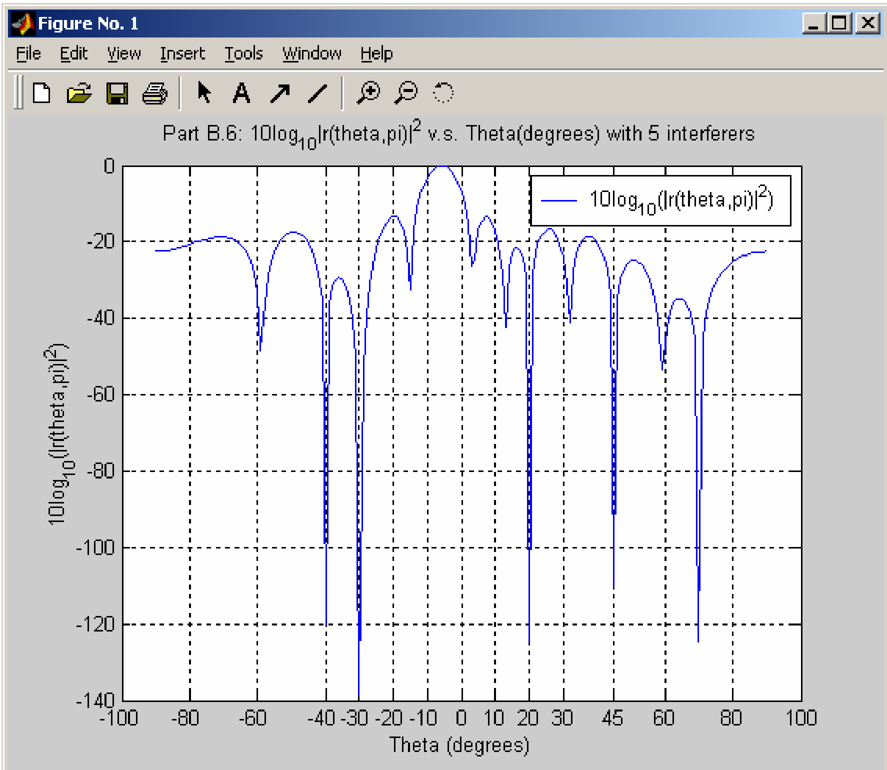


Figure No. 1
File Edit View Insert Tools Window Help
Part B.6: 10log10|r(theta,pi)|2 v.s. Theta(degrees) with 5 interferers
10log10(|r(theta,pi)|2)
0
-20
-40
-60
-80
-100
-120
-140
-100 -80 -60 -40 -30 -20 -10 0 10 20 30 45 60 80 100
Theta (degrees)